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A LOCAL-GLOBAL THEOREM ON PERIODIC MAPS

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Abstract. Let $\psi_1, \ldots, \psi_k$ be maps from $\mathbb{Z}$ to an additive abelian group with positive periods $n_1, \ldots, n_k$ respectively. We show that the function $\psi = \psi_1 + \cdots + \psi_k$ is constant if $\psi(x)$ equals a constant for $|S|$ consecutive integers x where $S = \{r/n_s : r = 0, \ldots, n_s - 1;\ s = 1, \ldots, k\}$; moreover, there are periodic maps $f_0, \ldots, f_{|S|-1} : \mathbb{Z} \to \mathbb{Z}$ only depending on S such that $\psi(x) = \sum_{r=0}^{|S|-1} f_r(x)\psi(r)$ for all $x \in \mathbb{Z}$. This local-global theorem extends a previous result [Math. Res. Lett. 11(2004), 187–196], and has various applications.

1. Introduction

In 1965 N. J. Fine and H. S. Wilf [FW] proved that for two real sequences $\{\alpha_j\}_{j\geqslant 0}$ and $\{\beta_j\}_{j\geqslant 0}$ with respective periods $m, n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, if $\alpha_j = \beta_j$ for all $0 \leqslant j < m+n-\gcd(m, n)$ (where $\gcd(m, n)$ denotes the greatest common divisor of m and n), then $\alpha_j = \beta_j$ for every $j = 0, 1, 2, \ldots$. This result has applications in combinatorics of finite words, see, e.g., [BB] and [R].

The author's investigation on covers of $\mathbb{Z}$ by residue classes led him to study periodic arithmetical maps in [S91], [S01], [S03] and [S04]. Periodic maps from $\mathbb{Z}$ to the complex field $\mathbb{C}$ include Dirichlet characters and the characteristic function of a residue class. In this paper we aim to establish the following general local-global theorem on periodic maps, which includes the Fine-Wilf result as a very particular case.

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Theorem 1.1. *Let G be any abelian group written additively, and let $\psi_1,\ldots,\psi_k$ be maps from $\mathbb{Z}$ to G with periods $n_1,\ldots,n_k\in\mathbb{Z}^+$ respectively. Set $\psi=\psi_1+\cdots+\psi_k$ and*

$$S(n_1,\ldots,n_k)=\bigcup_{s=1}^{k}\left\{\frac{r}{n_s}:\ r=0,\ldots,n_s-1\right\}. \tag{1.1}$$

(i) *There are periodic maps $f_0,\ldots,f_{|S(n_1,\ldots,n_k)|-1}:\mathbb{Z}\to\mathbb{Z}$ only depending on $S(n_1,\ldots,n_k)$ such that $\psi(x)=\sum_{0\leqslant r<|S(n_1,\ldots,n_k)|}f_r(x)\psi(r)$ for all $x\in\mathbb{Z}$. In particular, values of ψ are completely determined by the set $S(n_1,\ldots,n_k)$ and the initial values $\psi(0),\ldots,\psi(|S(n_1,\ldots,n_k)|-1)$.*

(ii) *ψ is constant if $\psi(x)$ equals a constant for $|S(n_1,\ldots,n_k)|$ $(\leqslant n_1+\cdots+n_k-k+1)$ consecutive integers x.*

Remark 1.1. Theorem 1.1(i) is completely new, and it seems that once $S=S(n_1,\ldots,n_k)$ is given those values of $f_0,\ldots,f_{|S|-1}$ at $0,\ldots,N-1$ are uniquely determined, where N stands for the least common multiple of $n_1,\ldots,n_k$ which is also the smallest common denominator of the rationals in S. When G is the additive group of a field whose characteristic does not divide any of the periods $n_1,\ldots,n_k$, Theorem 1.1(ii) was discovered by the author (cf. [S04]) in May 2002 via a method rooted in [S95], though he was unaware of the Fine-Wilf result at that time.

As for the cardinality of the set in (1.1), [S04, Remark 1.1] indicates that

$$|S(n_1,\ldots,n_k)|=\sum\nolimits_{d\mid n_s\ \text{for some}\ s=1,\ldots,k}\varphi(d), \tag{1.2}$$

where φ is the well-known Euler function. By the inclusion-exclusion principle in combinatorics, we also have

$$|S(n_1,\ldots,n_k)|=\sum_{\emptyset\neq I\subseteq\{1,\ldots,k\}}(-1)^{|I|-1}\gcd(n_s:\ s\in I), \tag{1.3}$$

because $|\bigcap_{s\in I}\{r/n_s:\ r=0,\ldots,n_s-1\}|=\gcd(n_s:\ s\in I)$ whenever $\emptyset\neq I\subseteq\{1,\ldots,k\}$.

In the case $k=2$, Theorem 1.1 yields the following consequence which is both stronger and more general than the Fine-Wilf result.

Corollary 1.1. *Let g and h be maps from $\mathbb{Z}$ to an additive abelian group G with positive periods m and n respectively. Then $\{g(x)-h(x):\ x\in\mathbb{Z}\}$ is contained in the subgroup of G generated by those $g(r)-h(r)$ with $0\leqslant r<m+n-\gcd(m,n)$; in particular, g and h are identical if $g(r)=h(r)$ for all $r=0,\ldots,m+n-\gcd(m,n)-1$.*

Proof. Since $|S(m,n)|=m+n-\gcd(m,n)$, it suffices to apply Theorem 1.1 with $\psi_1=g$ and $\psi_2=-h$. $\square$

Now we derive more consequences of Theorem 1.1.

Corollary 1.2. *Let $\{\psi_1(n)\}_{n\geqslant 0},\ldots,\{\psi_k(n)\}_{n\geqslant 0}$ be k periodic complex-valued sequences with periods $n_1,\ldots,n_k\in\mathbb{Z}^+$ respectively. Let $\psi(n)=\sum_{s=1}^k\psi_s(n)$ for $n=0,1,\ldots,$ and set*

$$L=\sum_{\emptyset\neq I\subseteq\{1,\ldots,k\}}(-1)^{|I|-1}\gcd(n_s:\ s\in I).$$

Then the sequence $\{\psi(n)\}_{n\geqslant 0}$ is a zero sequence if its initial L terms are zero. Also, the sequence $\{\psi(n)\}_{n\geqslant 0}$ is an integer sequence if its initial L terms are integers.

Proof. Observe that $L=|S(n_1,\ldots,n_k)|$ by (1.3). The first part follows from Theorem 1.1(ii) in the case $G=\mathbb{C}$, and it was realized by S. Cautis *et al.* [C] independent of the author's work in [S04]. The second part is a consequence of Theorem 1.1(i) in the case $G=\mathbb{C}$. $\square$

Corollary 1.3. *Let G be an additive abelian group, and let $c_1,\ldots,c_k\in G$ have orders $n_1,\ldots,n_k\in\mathbb{Z}^+$ respectively. For any $P_1(x),\ldots,P_k(x)\in\mathbb{Z}[x]$, the sum $P_1(x)c_1+\cdots+P_k(x)c_k$ vanishes for all $x\in\mathbb{Z}$ if it vanishes for $|S(n_1,\ldots,n_k)|$ consecutive integers x.*

Proof. For each $s=1,\ldots,k$, clearly n_s is a period of the map $\psi_s(x)=P_s(x)c_s$. So the desired result follows from Theorem 1.1(ii). $\square$

Corollary 1.4. *Let $\psi(x)=\sum_{s=1}^k\chi_s(P_s(x))$ for $x\in\mathbb{Z}$, where each χ_s is a Dirichlet character mod n_s and $P_1(x),\ldots,P_k(x)\in\mathbb{Z}[x]$. Then all those $\psi(x)$ with $x\in\mathbb{Z}$ are linear combinations of $\psi(0),\ldots,\psi(|S(n_1,\ldots,n_k)|-1)$ with integer coefficients.*

Proof. Since $\psi_s(x)=\chi_s(P_s(x))$ is a complex-valued function with period n_s, applying Theorem 1.1(i) we immediately get the required result. $\square$

Corollary 1.5. *Let $n_1,\ldots,n_k$ be positive integers, and let $\psi:\mathbb{Z}\to\mathbb{C}$ be a function with $\psi(0),\ldots,\psi(|S(n_1,\ldots,n_k)|)$ linearly independent over the field of rational numbers. Then ψ cannot be written in the form $\sum_{s=1}^k\psi_s$ where each ψ_s is a function from $\mathbb{Z}$ to $\mathbb{C}$ with period n_s.*

Proof. Let $S=S(n_1,\ldots,n_k)$. Now that there are no $c_0,\ldots,c_{|S|-1}\in\mathbb{Z}$ such that $\psi(|S|)=\sum_{r=0}^{|S|-1}c_r\psi(r)$, it suffices to apply Theorem 1.1(i) with $G=\mathbb{C}$. $\square$

Corollary 1.6. *Let $A=\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ be a finite system of residue classes. If $w_A(x):=|\{1\leqslant s\leqslant k:\ x\equiv a_s\ (\mathrm{mod}\ n_s)\}|$ lies in a residue class $a(\mathrm{mod}\ m)$ for $|S(n_1,\ldots,n_k)|$ consecutive integers x, then $w_A(\mathbb{Z})=\{w_A(x):\ x\in\mathbb{Z}\}\subseteq a(\mathrm{mod}\ m)$. In particular, A covers every integer an odd number of times if there are $|S(n_1,\ldots,n_k)|$ consecutive integers each of which is covered by A an odd number of times.*

Proof. Just apply Theorem 1.1(ii) with $G=\mathbb{Z}/m\mathbb{Z}$ and note that the characteristic function of $a_s(\mathrm{mod}\ n_s)$ has period n_s. $\square$

Corollary 1.7. *Let $A = \{a_s(\text{mod } n_s)\}_{s=1}^k$ $(k > 1)$ be a finite system of residue classes whose maximal moduli with respect to divisibility are distinct. Then, for any $a, b \in \mathbb{Z}$, we have*

$$\gcd\left(w_A(a) + b, \dots, w_A(a + |S(n_1, \dots, n_k)| - 1) + b\right) = 1. \tag{1.4}$$

Proof. Denote the left hand side of (1.4) by d. Then $w_A(a + r) \equiv -b \pmod{d}$ for all $0 \leqslant r < |S(n_1, \dots, n_k)|$, and hence $w_A(\mathbb{Z}) \subseteq -b(\text{mod } d)$ by Corollary 1.6. In view of [S05, Corollary 1.2], $w_A(\mathbb{Z})$ cannot be contained in any residue class with modulus greater than one. Therefore $d = 1$ and we are done. $\square$

2. Proof of Theorem 1.1

The old technique used to handle the special case of Theorem 1.1(ii) mentioned in Remark 1.1 is invalid for the general case. Thus, we have to work along a new line.

Let Ω denote the ring of all algebraic integers. Clearly all roots of unity belong to Ω.

Lemma 2.1. *Let $\psi(x) = \sum_{s=1}^k c_s \omega_s^x$ for $x \in \mathbb{Z}$, where $c_1, \dots, c_k \in \mathbb{C}$, and $\omega_1, \dots, \omega_k$ are roots of unity. Suppose that $\prod_{\zeta \in \{\omega_1, \dots, \omega_k\}}(x - \zeta) \in R[x]$ where R is a subring of Ω containing $\mathbb{Z}$. Then we have $\psi = \psi(0) f_0 + \dots + \psi(l-1) f_{l-1}$, where $l = |\{\omega_1, \dots, \omega_k\}|$, and $f_0, \dots, f_{l-1}$ are suitable periodic maps from $\mathbb{Z}$ to R only depending on the set $\{\omega_1, \dots, \omega_k\}$.*

Proof. Let $\zeta_1, \dots, \zeta_l$ be all the distinct roots of unity among $\omega_1, \dots, \omega_k$, and write

$$P(z) = \prod_{t=1}^{l}(z - \zeta_t) = z^l - a_1 z^{l-1} - \dots - a_{l-1} z - a_l,$$

where

$$a_j = (-1)^{j-1} \sum_{1 \leqslant i_1 < \dots < i_j \leqslant l} \zeta_{i_1} \cdots \zeta_{i_j} \in R \quad \text{for } j = 1, \dots, l.$$

Set $u_n = \sum_{t=1}^l c_t' \zeta_t^n$ for all $n \in \mathbb{Z}$, where $c_t' = \sum_{1 \leqslant s \leqslant k,\, \omega_s = \zeta_t} c_s$. Clearly $u_n = \sum_{s=1}^k c_s \omega_s^n = \psi(n)$. Also, $\{u_n\}_{n \in \mathbb{Z}}$ is a linear recurrence because

$$\begin{aligned} \sum_{j=1}^{l} a_j u_{n-j} &= \sum_{j=1}^{l} a_j \sum_{t=1}^{l} c_t' \zeta_t^{n-j} = \sum_{t=1}^{l} c_t' \zeta_t^{n-l} \sum_{j=1}^{l} a_j \zeta_t^{l-j} \\ &= \sum_{t=1}^{l} c_t' \zeta_t^{n-l} \left(\zeta_t^l - P(\zeta_t)\right) = u_n. \end{aligned}$$

If $u_{n-j} = f_0(n-j)u_0 + \cdots + f_{l-1}(n-j)u_{l-1}$ for all $j = 1, \ldots, l$ where $f_0(n-j), \ldots, f_{l-1}(n-j) \in R$, then

$$\begin{aligned} u_n &= \sum_{j=1}^{l} a_j \left(f_0(n-j)u_0 + \cdots + f_{l-1}(n-j)u_{l-1}\right) \\ &= \left(\sum_{j=1}^{l} a_j f_0(n-j)\right)u_0 + \cdots + \left(\sum_{j=1}^{l} a_j f_{l-1}(n-j)\right)u_{l-1}. \end{aligned}$$

Thus, by induction, for any $n = 0, \ldots, N-1$ we can write u_n in the form $f_0(n)u_0 + \cdots + f_{l-1}(n)u_{l-1}$, where N is the smallest positive integer with $\zeta_1^N = \cdots = \zeta_l^N = 1$, and $f_0, \ldots, f_{l-1}$ are suitable maps from $\{0, \ldots, N-1\}$ to R only depending on the set $\{\omega_1, \ldots, \omega_k\} = \{\zeta_1, \ldots, \zeta_l\}$. (Actually those f_r $(0 \leqslant r \leqslant l-1)$ can be constructed as follows: $f_r(r) = 1$, $f_r(n) = 0$ for $0 \leqslant n \leqslant l-1$ with $n \neq r$, and $f_r(n) = \sum_{j=1}^{l} a_j f_r(n-j)$ if $l \leqslant n < N$.)

If $x \in \mathbb{Z}$, and $x < 0$ or $x \geqslant N$, then we define $f_0(x), \ldots, f_{l-1}(x)$ to be $f_0(a), \ldots, f_{l-1}(a)$ respectively, where a is the least nonnegative residue of x modulo N, thus

$$\psi(x) = u_a = f_0(a)u_0 + \cdots + f_{l-1}(a)u_{l-1} = f_0(x)\psi(0) + \cdots + f_{l-1}(x)\psi(l-1).$$

In view of the above, we get the desired result. $\square$

The following lemma plays a central role in our proof of Theorem 1.1.

Lemma 2.2. *Let $\psi = \psi_1 + \cdots + \psi_k$ where each ψ_s $(1 \leqslant s \leqslant k)$ is a complex-valued function on $\mathbb{Z}$ with period $n_s \in \mathbb{Z}^+$. Then ψ can be written in the form $\sum_{0 \leqslant r < |S(n_1, \ldots, n_k)|} \psi(r) f_r$, where $f_0, \ldots, f_{|S(n_1, \ldots, n_k)|-1}$ are suitable periodic maps from $\mathbb{Z}$ to $\mathbb{Z}$ only depending on $S(n_1, \ldots, n_k)$.*

Proof. If $x \in \mathbb{Z}$ then

$$\begin{aligned} \psi(x) &= \sum_{s=1}^{k} \sum_{\substack{0 \leqslant a < n_s \\ n_s \mid x-a}} \psi_s(a) = \sum_{s=1}^{k} \frac{1}{n_s} \sum_{a=0}^{n_s-1} \psi_s(a) \sum_{r=0}^{n_s-1} e^{2\pi i \frac{r}{n_s}(x-a)} \\ &= \sum_{s=1}^{k} \sum_{r=0}^{n_s-1} \left(\frac{1}{n_s} \sum_{a=0}^{n_s-1} \psi_s(a) e^{-2\pi i a \frac{r}{n_s}}\right) \left(e^{2\pi i \frac{r}{n_s}}\right)^x. \end{aligned}$$

Observe that

$$\prod\nolimits_{\theta \in S(n_1, \ldots, n_k)} \left(x - e^{2\pi i \theta}\right) = \prod\nolimits_{d \mid n_s \text{ for some } s=1,\ldots,k} \Phi_d(x) \in \mathbb{Z}[x],$$

where $\Phi_d(x) = \prod_{0 \leqslant c < d,\, \gcd(c,d)=1} (x - e^{2\pi i c/d})$ is the dth cyclotomic polynomial. (That $\Phi_d(x) \in \mathbb{Z}[x]$ is well known, see, e.g., [IR, pp. 194-195].) Now it suffices to apply Lemma 2.1. $\square$

Remark 2.1. Let $m, n_1, \ldots, n_k \in \mathbb{Z}^+$, and let $f_0, \ldots, f_{|S(n_1,\ldots,n_k)|-1}$ be as in Lemma 2.2. For each $s = 1, \ldots, k$ let $\psi_s : \mathbb{Z} \to \mathbb{Z}$ be a map which has period n_s modulo m (i.e., $\psi_s(a) \equiv \psi_s(b) \pmod{m}$ whenever $a \equiv b \pmod{n_s}$). Let x be any integer. By Lemma 2.2 we have

$$\sum_{s=1}^{k} \psi_s'(x) = \sum\nolimits_{0\leqslant r<|S(n_1,\ldots,n_k)|} f_r(x) \sum_{s=1}^{k} \psi_s'(r),$$

where $\psi_s'(x) = \sum_{0\leqslant a<n_s,\, n_s|x-a} \psi_s(a)$. As $\psi_s'(x) \equiv \psi_s(x) \pmod{m}$ for each $s = 1, \ldots, k$, this yields that

$$\psi(x) \equiv \sum\nolimits_{0\leqslant r<|S(n_1,\ldots,n_k)|} f_r(x)\psi(r) \pmod{m},$$

where $\psi = \psi_1 + \cdots + \psi_k$. For $a \in \mathbb{Z}$ let $\bar{a}$ denote the residue class $a(\text{mod } m)$ in the ring $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Then we have

$$\sum_{s=1}^{k} \overline{\psi_s(x)} = \overline{\psi(x)} = \sum\nolimits_{0\leqslant r<|S(n_1,\ldots,n_k)|} f_r(x)\overline{\psi(r)}.$$

This shows that Theorem 1.1(i) holds when G is the cyclic group $\mathbb{Z}_m$.

Proof of Theorem 1.1. (i) Without any loss of generality, we simply let G coincide with its subgroup generated by the finite set

$$\{\psi_s(x) :\ x = 0, \ldots, n_s - 1;\ s = 1, \ldots, k\}.$$

Since G is finitely generated, there are $m_1, \ldots, m_l \in \mathbb{Z}^+$ and $n \in \{0, 1, \ldots\}$ such that G is isomorphic to the direct sum $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_l} \oplus \mathbb{Z}^n$. Let us identify G with $G_1 \oplus \cdots \oplus G_{l+n}$, where $G_t = \mathbb{Z}_{m_t}$ for $t = 1, \ldots, l$, and $G_{l+1} = \cdots = G_{l+n} = \mathbb{Z}$.

Let $f_0, \ldots, f_{|S(n_1,\ldots,n_k)|-1}$ be as in Lemma 2.2, and let x be any integer. For $s = 1, \ldots, k$ we write $\psi_s(x)$ in the vector form

$$\langle \psi_{s,\,1}(x), \ldots, \psi_{s,\,l+n}(x)\rangle,$$

where $\psi_{s,\,t}(x) \in G_t$ for $t = 1, \ldots, l + n$. Set $\psi^{(t)} = \sum_{s=1}^{k} \psi_{s,\,t}$ for $t = 1, \ldots, l + n$. Since $\psi_{s,\,t} : \mathbb{Z} \to G_t$ also has period n_s, we have

$$\psi^{(t)}(x) = \sum\nolimits_{0\leqslant r<|S(n_1,\ldots,n_k)|} f_r(x)\psi^{(t)}(r)$$

by Lemma 2.2 and Remark 2.1. Therefore,

$$\begin{aligned}\psi(x) =& \langle \psi^{(1)}(x), \ldots, \psi^{(l+n)}(x)\rangle \\ =& \sum\nolimits_{0\leqslant r<|S(n_1,\ldots,n_k)|} f_r(x)\langle \psi^{(1)}(r), \ldots, \psi^{(l+n)}(r)\rangle \\ =& \sum\nolimits_{0\leqslant r<|S(n_1,\ldots,n_k)|} f_r(x)\psi(r).\end{aligned}$$

This proves the first part of Theorem 1.1.

(ii) Now suppose that $\psi(a+r)=c$ for all $r=0,\ldots,|S(n_1,\ldots,n_k)|-1$, where $a\in\mathbb{Z}$ and $c\in G$. For $x\in\mathbb{Z}$ let $\psi^*(x)=\psi_s(a+x)$ for $1\leqslant s<k$, $\psi_k^*(x)=\psi_k(a+x)-c$ and $\psi^*(x)=\psi_1^*(x)+\cdots+\psi_k^*(x)=\psi(a+x)-c$. By the first part of Theorem 1.1, the range of ψ^* is contained in the subgroup of G generated by $\{\psi^*(r):\ 0\leqslant r<|S(n_1,\ldots,n_k)|\}=\{0\}$. Thus $\psi(a+x)-c=\psi^*(x)=0$ for all $x\in\mathbb{Z}$. This concludes our proof. $\square$

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